B. Math. III – Mid-Term Examination

Introduction to Differential Geometry

September 10, 2014

1. Define the vector product of two vectors in $\mathbb{R}^3$. Prove that for vectors u, v, x, y in $\mathbb{R}^3$, the dot product $(u \times v) \cdot (x \times y)$ equals the determinant of the matrix $\begin{bmatrix} u \cdot x & v \cdot x \\ u \cdot y & v \cdot y \end{bmatrix}$.

2. (*Viviani's curve*) Show that $\gamma(t) = (\cos^2 t - \frac{1}{2}, \sin t \cos t, \sin t)$ is a parametrisation of the curve of intersection of circular cylinder of radius $\frac{1}{2}$ and axis the z-axis with sphere of radius 1 and center $(-\frac{1}{2}, 0, 0)$.

3. Define torsion τ of a regular curve in $\mathbb{R}^3$ having nowhere zero curvature. Prove that τ is zero if and only if the curve lies in a plane.

4. Compute the torsion τ and curvature κ of the Viviani's curve given above and verify that:

$$\frac{\tau}{\kappa} = \frac{d}{ds}\left(\frac{\dot{\kappa}}{\tau\kappa^2}\right).$$

5. Show that the ellipse

$$\gamma(t) = (a\cos(t), b\sin(t)),$$

where a and b are positive constants, is a simple closed curve and compute the area of its interior.